Vol. 05, No. 02, pp. 55 –62 (2024)
ISSN: 2709-6904

JOURNAL OF CIVIL ENGINEERING FRONTIERS

www.jocivilef.org

First Principles Variational Formulation and Analytical Solutions for Third Order Shear Deformable Beams with Simple Supports using Fourier Series Method

Charles Chinwuba Ike[1]*, Benjamin Okwudili Mama[2], Onyedikachi Aloysius Oguaghamba[2]

[1]*Department of Civil Engineering, Enugu State University of Science and Technology, Agbani, Enugu State, Nigeria, charles.ike@esut.edu.ng*
[2]*Department of Civil Engineering, University of Nigeria, Nsukka, Enugu State, Nigeria, (benjamin.mama@unn.edu.ng, aloysius.oguaghamba@unn.edu.ng)*

Abstract

The Fourier series method for solving bending problems of third order shear deformable beams (TSBD) is presented in this paper. The theory accounts for transverse shear deformation and is suitable for moderately thick beams. Transverse shear stress free conditions are valid at the beam surfaces for TSDB. The field equations are two coupled ordinary differential equations in terms of two unknown displacement functions – transverse deflection $w(x)$ and warping function $\varphi(x)$. For simply supported ends considered, the loading and unknown functions $w(x)$ and $\varphi(x)$ are represented in Fourier series that satisfies the boundary conditions. The problem is reduced to a system of algebraic equations in terms of the Fourier coefficients w_n and φ_n, which is solved to obtain w_n and φ_n. Axial and transverse displacements fields, axial bending stress σ_{xx} and transverse shear stress τ_{xz} are then determined for the cases of point load at midspan, uniformly and linearly distributed loads over the span. The results are identical with previous results obtained by other scholars who used Ritz variational methods and other mathematical tools. For moderately thick beams under uniform load with l/h = 4, results obtained for σ_{xx} is 0.516% greater than the exact result by Timoshenko and Goodier while for thick beams under uniform load with l/h = 2, the result for σ_{xx} is 1.96% greater than the exact result by Timoshenko and Goodier. Similar acceptable variation was obtained for σ_{xx} for beams under linearly distributed load with the variations being less than 2% for l/h = 2. However, unacceptable variations of 68.37% were found for σ_{xx} in thick beams under point load, for l/h = 2.0. The variation for σ_{xx} however reduced to 12.513% for moderately thick beams under point load for l/h =4.

Keywords: *Third order shear deformable beam theory, Fourier series method, warping function, deflection function*

Received: June 3, 2024 / Accepted: October 10, 2024 / Online: October 12, 2024

I. INTRODUCTION

Beams are flexural structures (with application to all fields of engineering) which are usually submitted to transverse point and distributed loads acting on the span. In the process axial bending stresses and transverse shear stresses, and axial and transverse displacements are developed in the beam. The mechanics of beams is affected by the ratio of the transverse dimension to the longitudinal dimension; and this ratio determines the categorization as thin, moderately thick and thick beams.

The well-known Euler-Bernoulli theory (EBT) was developed for thin beams using the Navier's (Bernoulli's) hypothesis [1 – 4]. It has been found to give satisfactory results for cases where transverse shear deformations do not make an important contribution to the behavior [1 – 4].

Other studies, formulations and theories derived to take consideration of transverse shear deformations are by Timoshenko [5], Mindlin, Shimpi, Levinson [6], Reissner, Sayyad, Shimpi et al [7], Ghugal and Sharma [8], Ambartsumyan [9], Kruszewski [10], Akavci [11], Reddy [12], Timoshenko and Goodier [13].

doi:10.38094/jocef50286

Timoshenko's beam theory (TBT) and Mindlin beam theories are first order shear deformation theories (FSDT) which assume that the transverse shear strain is constant in the beam transverse dimension; thus violating the transverse shear stress force conditions at the beam surfaces ($z = \pm h/2$). Shear correction factors are introduced in the TBT to obtain transverse shear stresses that agree with energy principles.

Ike [14] used the Ritz method to solve the bending problems of third order shear deformable beams subjected to uniformly distributed loading over the entire span. Ike and Oguaghamba [15] applied the Fourier series method to solve flexural problems of thick beams modelled using trigonometric shear deformation theory.

Ike [16] presented variational formulation and analytical solutions to the flexural problems of moderately thick beams modelled using Timoshenko theory. Onah et al [17] used classical mathematical methods to obtain closed form solutions to the elastic stability problems of moderately thick beams. Ike et al [18] used the Laplace transformation method to obtain the buckling loads of moderately thick beams for various end support conditions.

Ghugal [19] solved in closed form the bending problem of thick beams using plane elasticity methods. Other important contributions to the studies of moderately thick beams include Ghugal and Shimpi [20], Heyliger and Reddy [21], Ghugal and Dahake [22] and Sayyad [23, 24].

Reissner beam theory is a stress-based theory while Levinson and Reddy are third higher order shear deformation theories. Higher order shear deformation theories have been presented. Refined beam theories (RBT) were formulated by Shimpi, Shimpi et al [7], and Ghugal and Sharma [8]. Timoshenko and Goodier [13] solved thick beam bending problems using elasticity theory.

This paper presents the Fourier series method for solving third order shear deformable beam bending problems. Three cases of loading considered are: point load at midspan, uniformly distributed load, and linearly distributed load over the span.

Limitations of the proposed method

The proposed Fourier series method is limited to beams that are simply supported at the ends. This enables the Fourier sine series to be a suitable displacement shape function since the displacement boundary conditions are satisfied at the simply supported ends. It is also applicable to specific loading conditions where the loads could be expressed using Fourier series theory.

II. The Third Order Shear Deformation Beam Theory

The thick beam bending problem due to transverse load as shown in Figure 1 is considered. The beam's material properties are E = 210GPa, $\mu = 0.30$, $\rho = 7800\text{kg/m}^3$. E is the Young's modulus of elasticity, ρ is the mass density, μ is the Poisson's ratio.

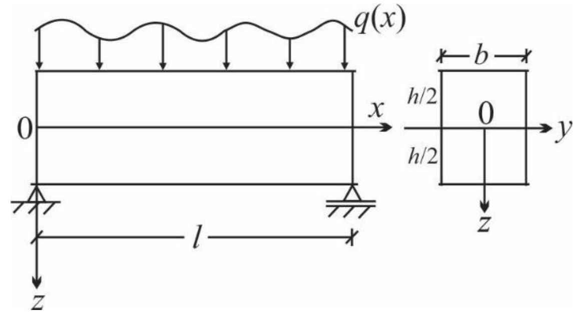

Fig. 1. Thick beam carrying transverse load $q(x)$

The beam domain is: $0 \le x \le l$, $-b/2 \le y \le b/2$, $-h/2 \le z \le h/2$

h is the depth (thickness) of the beam, b is the width, and l is the span of the beam.

A. Assumptions

The assumptions are

(i) the axial displacement is composed of a bending and shear deformation component

(ii) the transverse displacement $w(x)$ depends only on the x coordinate

(iii) the beam is subjected to lateral loads only

(iv) the relations between the stresses and strains are one-dimensional.

B. Displacement

The displacement components are expressed by:

$$u(x,z) = -z\frac{dw}{dx} + \left(z - \frac{4z^3}{3h^2}\right)\varphi(x) \quad (1)$$

$$w(x,z) = w(x) \quad (2)$$

The normal strains ε_{xx} and shear strain γ_{xz} are found using the kinematic relations as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{d^2w}{dx^2} + \left(z - \frac{4z^3}{3h^2}\right)\varphi'(x) \quad (3)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\varphi(x)\left(1 - \frac{4z^2}{h^2}\right) \quad (4)$$

The normal stress σ_{xx} and shear stress τ_{xz} are

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(-z\frac{d^2w}{dx^2} + \left(z - \frac{4z^3}{3h^2}\right)\varphi'(x)\right) \quad (5)$$

$$\tau_{xz} = G\gamma_{xz} = G\left(1 - \frac{4z^2}{h^2}\right)\varphi(x) \quad (6)$$

G is the shear modulus of the beam material.

The potential energy functional Π is expressed by:

$$\Pi = \frac{1}{2}\int_{-b/2}^{b/2}\int_{0}^{l}\int_{-h/2}^{h/2}(\sigma_{xx}\varepsilon_{xx} + \tau_{xz}\gamma_{xz})dzdxdy - \int_{0}^{l} q(x)w(x)dx \quad \ldots(7)$$

Thus,

$$\Pi = \frac{1}{2}\int_{0}^{l}\int_{-h/2}^{h/2}\left\{bE\left(-z\frac{d^2w}{dx^2} + \left(z - \frac{4z^3}{3h^2}\right)\frac{d\varphi}{dx}\right)^2 + bG\left(1 - \frac{4z^2}{h^2}\right)^2 (\varphi(x))^2\right\}dxdz - \int_{0}^{l} q(x)w(x)dx \quad (8)$$

Simplifying,

$$\Pi = \frac{1}{2}\int_{0}^{l}\left\{bE\int_{-h/2}^{h/2} z^2\left(\frac{d^2w}{dx^2}\right)^2 + \left(z - \frac{4z^3}{3h^2}\right)^2\left(\frac{d\varphi}{dx}\right)^2 - 2z\left(z - \frac{4z^3}{3h^2}\right)\frac{d^2w}{dx^2}\frac{d\varphi}{dx} + Gb\left(1 - \frac{4z^2}{h^2}\right)^2 \left(\varphi^2(x)\right)\right\}dxdz - \int_{0}^{l} q(x)w(x)dx \quad (9)$$

Thus,

$$\Pi = \frac{1}{2}\int_{0}^{l}\left(F_1\left(\frac{d^2w}{dx^2}\right)^2 - 2F_2\frac{d^2w}{dx^2}\frac{d\varphi}{dx} + F_3\left(\frac{d\varphi}{dx}\right)^2 + F_4\varphi^2(x) - 2q(x)w(x)\right)dx \quad (10)$$

$$\Pi = \frac{1}{2}\int_{0}^{l}\left(F_1 w_{xx}^2 - 2F_2 w_{xx}\varphi_x + F_3(\varphi_x(x))^2 + F_4(\varphi(x))^2 - 2q(x)w(x)\right)dx \quad (11)$$

$$F_1 = \int_{-h/2}^{h/2} Ebz^2 dz = EI \quad (12)$$

$$F_2 = Eb\int_{-h/2}^{h/2} z\left(z - \frac{4z^3}{3h^2}\right)dz = \frac{Ebh^3}{15} = 0.8EI \quad (13)$$

$$F_3 = Eb\int_{-h/2}^{h/2}\left(z - \frac{4z^3}{3h^2}\right)^2 dz = \frac{17Ebh^3}{315} = \frac{204EI}{315} \quad (14)$$

$$F_4 = Gb\int_{-h/2}^{h/2}\left(1 - \frac{4z^2}{3h^2}\right)^2 dz = \frac{8}{15}Gbh = \frac{4Ebh}{15(1+\mu)} = \frac{4EA}{15(1+\mu)} = \frac{4E}{15(1+\mu)}\frac{I}{12h^2} = \frac{4EA}{15(1+\mu)} \quad (15a)$$

$$F_4 = \frac{EI}{45(1+\mu)h^2} = \frac{4E}{15(1+\mu)}\frac{I}{12h^2} \quad (15b)$$

III. Euler-Lagrange Equations of Equilibrium

The Euler-Lagrange equations of equilibrium are obtained using:

$$\frac{\partial F}{\partial \varphi(x)} - \frac{\partial}{\partial x}\frac{\partial F}{\partial \varphi_x(x)} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial \varphi_{xx}(x)} = 0 \quad (16)$$

$$\frac{\partial F}{\partial w(x)} - \frac{\partial}{\partial x}\frac{\partial F}{\partial w_x} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial w_{xx}} = 0 \quad (17)$$

Then,

$$F_1\frac{d^4w(x)}{dx^4} - F_2\frac{d^3\varphi(x)}{dx^3} = q(x) \quad (18)$$

$$F_2\frac{d^3w(x)}{dx^3} - F_3\frac{d^2\varphi(x)}{dx^2} + F_4\varphi(x) = 0 \quad (19)$$

Then,

$$EI\frac{d^4w(x)}{dx^4} - 0.8EI\frac{d^3\varphi(x)}{dx^3} = q(x) \quad (20)$$

$$0.8EI\frac{d^3w(x)}{dx^3} - \frac{204EI}{315}\frac{d^2\varphi(x)}{dx^2} + \frac{EI}{45(1+\mu)h^2}\varphi(x) = 0 \quad (21)$$

For the considered problem, the boundary conditions are:

$$w(x=0) = w(x=l) = 0 \quad (22a)$$

$$w_{xx}(x=0) = w_{xx}(x=l) = 0 \quad (22b)$$

$$\varphi_x(x=0) = \varphi_x(x=l) = 0 \quad (22c)$$

Thus, suitable Fourier series functions for $w(x)$ and $\varphi(x)$ are

$$w(x) = \sum_{n=1}^{\infty} w_n \sin\frac{n\pi x}{l} \quad (23a)$$

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \cos\frac{n\pi x}{l} \quad (23b)$$

where w_n and φ_n are amplitudes of $w(x)$ and $\varphi(x)$.

Then, the system of domain equations become:

$$EI\frac{d^4}{dx^4}\sum_{n=1}^{\infty} w_n \sin\frac{n\pi x}{l} - 0.8EI\frac{d^3}{dx^3}\sum_{n=1}^{\infty}\varphi_n\cos\frac{n\pi x}{l} = \sum_{n=1}^{\infty} q_n \sin\frac{n\pi x}{l} \quad (24)$$

$$0.8EI\frac{d^3}{dx^3}\sum_{n=1}^{\infty} w_n \sin\frac{n\pi x}{l} - \frac{204EI}{315}\frac{d^2}{dx^2}\sum_{n=1}^{\infty}\varphi_n\cos\frac{n\pi x}{l} + \frac{EI}{45(1+\mu)h^2}\sum_{n=1}^{\infty}\varphi_n\cos\frac{n\pi x}{l} = 0 \quad (25)$$

$$\sum_{n=1}^{\infty}\left\{EI\left(\frac{n\pi}{l}\right)^4 w_n - 0.8EI\left(\frac{n\pi}{l}\right)^3\varphi_n\right\}\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} q_n \sin\frac{n\pi x}{l} \quad \ldots(26)$$

$$\sum_{n=1}^{\infty}\left\{-0.8EI\left(\left(\frac{n\pi}{l}\right)^3 w_n + \left(\frac{204EI}{315}\left(\frac{n\pi}{l}\right)^2 + \right.\right.\right.$$

$$\left.\left.\frac{EI}{45(1+\mu)h^2}\right)\varphi_n\right\}\cos\frac{n\pi x}{l} = 0 \quad (27)$$

Orthogonalizing, we have:

$$\sum_{n=1}^{\infty}\left\{EI\left(\frac{n\pi}{l}\right)^4 w_n - 0.8EI\left(\frac{n\pi}{l}\right)^3\varphi_n\right\}I_{mn} = \sum_{n=1}^{\infty}q_n I_{mn} \quad (28)$$

$$I_{mn} = \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx \quad (29)$$

$$\sum_{n=1}^{\infty}\left\{-0.8EI\left(\frac{n\pi}{l}\right)^3 w_n + \left(\frac{204EI}{315}\left(\frac{n\pi}{l}\right)^2 + \frac{EI}{45(1+\mu)h^2}\right)\varphi_n\right\}I^*_{mn} = 0 \quad (30)$$

$$I^*_{mn} = \int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx \quad (31)$$

Then, we have,

$$\begin{pmatrix} EI\left(\frac{n\pi}{l}\right)^4 & -0.8EI\left(\frac{n\pi}{l}\right)^3 \\ -0.8EI\left(\frac{n\pi}{l}\right)^3 & \left(\frac{204}{315}EI\left(\frac{n\pi}{l}\right)^2 + \frac{EI}{45(1+\mu)h^2}\right)\end{pmatrix}\begin{pmatrix} w_n \\ \varphi_n\end{pmatrix} = \begin{pmatrix} q_n \\ 0\end{pmatrix} \quad \ldots(32)$$

Dividing by EI gives:

$$\begin{pmatrix} \left(\frac{n\pi}{l}\right)^4 & -0.8EI\left(\frac{n\pi}{l}\right)^3 \\ -0.8\left(\frac{n\pi}{l}\right)^3 & \left(\frac{204}{315}\left(\frac{n\pi}{l}\right)^2 + \frac{1}{45(1+\mu)h^2}\right)\end{pmatrix}\begin{pmatrix} w_n \\ \varphi_n\end{pmatrix} = \begin{pmatrix} \frac{q_n}{EI} \\ 0\end{pmatrix} \quad \ldots(33)$$

Solving,

$$w_n = \frac{\Delta_{11}}{\Delta_{00}} \quad (34a)$$

$$\varphi_n = \frac{\Delta_{22}}{\Delta_{00}} \quad (34b)$$

$$\Delta_{11} = \begin{vmatrix} \frac{q_n}{EI} & -0.8\left(\frac{n\pi}{l}\right)^3 \\ 0 & \left(\frac{204}{315}\left(\frac{n\pi}{l}\right)^2 + \frac{1}{45h^2(1+\mu)}\right)\end{vmatrix} \quad (35)$$

$$\Delta_{22} = \begin{vmatrix} \left(\frac{n\pi}{l}\right)^4 & \frac{q_n}{EI} \\ -0.8\left(\frac{n\pi}{l}\right)^3 & 0\end{vmatrix} \quad (36)$$

$$\Delta_{00} = \begin{vmatrix} \left(\frac{n\pi}{l}\right)^4 & -0.8\left(\frac{n\pi}{l}\right)^3 \\ -0.8\left(\frac{n\pi}{l}\right)^3 & \left(\frac{204}{315}\left(\frac{n\pi}{l}\right)^2 + \frac{1}{45h^2(1+\mu)}\right)\end{vmatrix} \quad (37)$$

$$w_n = \frac{\frac{q_n}{EI}\left(\frac{204}{315}\left(\frac{n\pi}{l}\right)^2 + \frac{1}{45h^2(1+\mu)}\right)}{\left(\frac{n\pi}{l}\right)^4\left(\frac{204}{315}\left(\frac{n\pi}{l}\right)^2 + \frac{1}{45h^2(1+\mu)}\right) - \left(0.8\left(\frac{n\pi}{l}\right)^3\right)^2} \quad \ldots(38)$$

$$\varphi_n = \frac{\frac{q_n}{EI}\left(0.8\left(\frac{n\pi}{l}\right)^3\right)}{\left(\frac{n\pi}{l}\right)^4\left(\frac{204}{315}\left(\frac{n\pi}{l}\right)^2 + \frac{1}{45h^2(1+\mu)}\right) - \left(0.8\left(\frac{n\pi}{l}\right)^3\right)^2} \quad \ldots(39)$$

For sinusoidal load $q(x) = q_0\sin\frac{\pi x}{l}$

$$q_n = q_0 \quad (40)$$

For uniformly distributed load, $q(x) = q_1$

$$q_n = \frac{4q_1}{n\pi} \qquad n = 1, 3, 5, 7, \ldots \quad (41a)$$

$$q_n = 0 \quad n = 2, 4, 6, 8, \ldots \quad (41b)$$

For linearly distributed load, $q(x) = q_2\, x/l$

$$q_n = \frac{2q_2}{l^2}\int_0^l x\sin\frac{n\pi x}{l}dx \quad (42)$$

For point load $q(x) = Q$ at $x = \varepsilon$,

$$q_n = \frac{2Q}{l}\sin\frac{n\pi\varepsilon}{l} \quad (43a)$$

When $x = \varepsilon = l/2$,

$$q_n = \frac{2Q}{l}\sin\frac{n\pi}{2} \quad (43b)$$

A. Axial displacement, u

The axial displacement field is found as:

$$u(x,z) = \sum_{n=1}^{\infty}\left(-z\left(\frac{n\pi}{l}\right)w_n + \left(z - \frac{4z^3}{3h^2}\right)\varphi_n\right)\cos\frac{n\pi x}{l} \quad (44)$$

The maximum transverse displacement w occurs at $x = l/2$, $z = 0$, and is

$$w\left(x = l/2, z = 0\right) = \sum_{n=1}^{\infty} w_n\sin\frac{n\pi}{2} \quad (45)$$

The axial bending stress field $\sigma_{xx}(x,z)$ is

$$\sigma_{xx}(x,z) = E\sum_{n=1}^{\infty}\left(z\left(\frac{n\pi}{l}\right)^2 w_n + \left(z - \frac{4z^3}{3h^2}\right)\frac{n\pi}{l}\varphi_n\right)\sin\frac{n\pi x}{l} \quad \ldots(46)$$

Then,

$$\sigma_{xx}\left(x = l/2, z = \pm h/2\right) = E\sum_{n=1}^{\infty}\left(\pm\frac{h}{2}\left(\frac{n\pi}{l}\right)^2 w_n + g\left(\pm\frac{h}{2}\right)\frac{n\pi}{l}\varphi_n\right)\sin\frac{n\pi}{2} \quad (47)$$

where $g(z) = z - \frac{4z^3}{3h^2}$

The transverse shear stress τ_{xz} is found as

$$\tau_{xz} = \sum_{n=1}^{\infty} G\left(1 - \frac{4z^2}{h^2}\right)\varphi_n \cos\frac{n\pi x}{l} \quad (48)$$

Then,

$$\tau_{xz}(x = 0, z = 0) = \sum_{n=1}^{\infty} G\varphi_n \quad (49)$$

IV. Results

Stresses and displacements at points on the thick beam subjected to uniformly distributed load, linearly distributed load, and point load at the center are shown in Tables 1 – 6. The non-dimensional displacements $\bar{u}, \bar{w}$ and stresses $\bar{\sigma}_{xx}, \bar{\tau}_{xz}$ used in the tables are defined as follows:

$$w = \bar{w}\frac{p_0 l}{10Eb}\left(\frac{l}{h}\right)^3 \quad (50a)$$

$$\sigma_{xx} = \bar{\sigma}_{xx}\left(\frac{p}{b}\right) \quad (50b)$$

$$\tau_{xz} = \bar{\tau}_{xz}\left(\frac{p}{b}\right) \quad (50c)$$

TABLE I. VALUES OF AXIAL AND TRANSVERSE DISPLACEMENT AT GIVEN POINTS ON A THICK ISOTROPIC BEAM UNDER UNIFORMLY DISTRIBUTED LOAD FOR DIFFERENT ASPECT RATIOS

l/h	Reference/Method	$\bar{u}(x = l, z = \pm h/2)$	% Difference	$\bar{w}(x = l/2, z = 0)$	% Difference
2	Present	2.245	2.045	2.532	3.221
	Reddy [12]	2.245	2.045	2.532	3.221
	Timoshenko [5]	2.000	–9.091	2.538	3.465
	Ambartsumyan [9]	–	–	2.357	–3.913
	Kruszewski [10]	–	–	2.215	2.527
	Akavci [11]	–	–	2.523	2.853
	Euler-Bernoulli	2.000	–9.091	1.563	–36.282
	Ike [14] (Ritz method)	2.245	2.682	2.532	3.221
	Timoshenko and Goodier [13]	2.200	0	2.453	0
4	Present	16.504	4.456	1.806	1.176
	Reddy [12]	16.504	4.456	1.806	1.176
	Timoshenko [5]	16.000	1.265	1.806	1.176
	Ambartsumyan [9]	–	–	1.762	–1.288
	Kruszewski [10]	–	–	1.805	1.120
	Akavci [11]	–	–	1.804	1.064
	Euler-Bernoulli	16.000	1.265	1.563	–12.437
	Ike [14] (Ritz method)	16.504	4.456	1.806	1.176
	Timoshenko and Goodier [13]	15.800	0	1.785	0
10	Present	251.27	0.709	1.602	0.25
	Reddy [12]	251.27	0.709	1.602	0.25
	Timoshenko [5]	250.00	0.200	1.602	0.25
	Ambartsumyan [9]	–	–	1.595	–0.187
	Kruszewski [10]	–	–	1.602	0.25
	Akavci [11]	–	–	1.601	0.187
	Euler-Bernoulli	250.00	0.200	1.563	–2.190
	Ike [14] (Ritz method)	251.27	0.709	1.602	0.25
	Timoshenko and Goodier [13]	249.50	0	1.598	0

TABLE II. NON-DIMENSIONAL AXIAL STRESSES $\sigma_{xx}(x = l/2, z = h/2)$ AND TRANSVERSE SHEAR STRESSES $\tau_{xz}(x = 0, z = 0)$ FOR THICK ISOTROPIC BEAM CARRYING UNIFORMLY DISTRIBUTED LOAD FOR VARIOUS ASPECT RATIOS

l/h	Reference/Theory	$\bar{\sigma}_{xx}$	% Diff	$\bar{\tau}_{xz}^{CR}$	% Diff	$\bar{\tau}_{xz}^{EE}$	% Diff
2	Present	3.261	1.960	1.415	–5.667	1.262	–15.867
	Reddy [12]	3.261	1.960	1.415	–5.667	1.262	–15.867
	Timoshenko [5]	3.000	–6.25	0.984	–34.40	1.477	–1.533
	Ambartsumyan [9]	3.210	0.312	1.156	–22.93	–	–
	Kruszewski [10]	3.261	1.906	1.333	–11.13	–	–
	Akavci [11]	3.253	1.656	1.397	–6.866	–	–
	Euler-Bernoulli	3.000	–6.25	–	–	1.477	–1.533
	Ike [14]	3.261	1.960	1.415	–5.667	1.262	–15.867
	Timoshenko and Goodier [13]	3.200	0	1.500	0	1.500	0
4	Present	12.263	0.516	2.908	–3.067	2.795	–6.833
	Reddy [12]	12.263	0.516	2.908	–3.067	2.795	–6.833
	Timoshenko [5]	12.000	–1.693	1.969	–34.367	2.953	–1.567
	Ambartsumyan [9]	12.212	0.098	2.389	–20.36	–	–
	Kruszewski [10]	12.262	0.508	2.836	–5.466	–	–
	Akavci [11]	12.254	0.442	2.882	–3.933	–	–
	Euler-Bernoulli	12.00	–1.693	–	–	2.953	–1.567
	Ike [14]	12.263	0.516	2.908	–3.067	–	–
	Timoshenko and Goodier [13]	12.20	0	3.000	0	3.000	0
10	Present	75.268	0.090	7.361	–1.853	7.304	–2.61
	Reddy [12]	75.268	0.090	7.361	–1.853	7.304	–2.61
	Timoshenko [5]	75.00	–0.264	4.922	–34.373	7.383	–1.56
	Ambartsumyan [9]	75.216	0.021	6.066	–19.12	–	–
	Kruszewski [10]	75.266	0.087	7.328	–2.293	–	–
	Akavci [11]	75.259	0.078	7.312	–2.506	–	–
	Euler-Bernoulli	75.00	–0.264	–	–	7.383	–1.56
	Ike [14]	75.268	0.090	7.361	–1.853	–	–
	Timoshenko and Goodier [13]	75.20	0	7.500	0	7.500	0

TABLE III. NON-DIMENSIONAL AXIAL DISPLACEMENTS $\bar{u}(x = l, z = h/2)$ AND TRANSVERSE DISPLACEMENTS $\bar{w}(x = l/2, z = 0)$ FOR THICK ISOTROPIC BEAMS CARRYING POINT LOAD AT THE CENTER OF SPAN FOR VARIOUS ASPECT RATIOS

l/h	Theory/Method/Reference	$\bar{u}$	% Difference	$\bar{w}$	% Difference
2	Present method	3.2611	–	4.3399	7.899
	Reddy [12]	3.2611	–	4.3399	7.899

	Timoshenko [5]	3.0001	–	4.4198	9.978
	Euler-Bernoulli	3.0001	–	2.5000	–37.792
	Timoshenko and Goodier [13]	–	–	4.0188	0
4	Present method	25.5263	–	2.9726	2.0635
	Reddy [12]	25.5263	–	2.9726	2.0635
	Timoshenko [5]	24.0007	–	2.9799	2.3142
	Euler-Bernoulli	24.0007	–	2.5000	–14.1631
	Timoshenko and Goodier [13]	–	–	2.9125	0
10	Present method	376.3385	–	2.5765	0.292
	Reddy [12]	376.3385	–	2.5765	0.292
	Timoshenko [5]	375.0122	–	2.5768	0.304
	Euler-Bernoulli	375.0109	–	2.5000	–2.686
	Timoshenko and Goodier [13]	–	–	2.5690	0

TABLE IV. NON-DIMENSIONAL AXIAL BENDING STRESSES $\bar{\sigma}_{xx}$ AT $(x = l/2, z = h/2)$, TRANSVERSE SHEAR STRESSES $\bar{\tau}_{xz}^{CR}$ $(x = 0, z = 0)$, $\bar{\tau}_{xz}^{EE}(x = 0, z = 0)$ FOR POINT LOAD AT THE MIDSPAN OF THICK ISOTROPIC BEAM FOR VARIOUS ASPECT RATIOS (l/h).

l/h	Theory/Reference	$\bar{\sigma}_{xx}$	% Diff	τ_{xz}^{CR}	% Diff	τ_{xz}^{EE}	% Diff
2	Present method	9.3469	68.571	1.5059	–	1.4290	–
	Reddy [12]	9.3469	68.571	1.5059	–	1.4290	–
	Timoshenko [5]	5.9065	6.523	1.0244	–	1.5367	–
	Euler-Bernoulli	5.9065	6.523	–	–	1.5367	–
	Timoshenko and Goodier [13]	5.5448	0	–	–	–	–
4	Present method	28.6790	12.518	3.0319	1.063	2.9284	–
	Reddy [12]	28.6770	12.513	3.0319	1.063	2.9284	–
	Timoshenko [5]	23.6261	–7.311	2.0489	–31.703	3.0733	–
	Euler-Bernoulli	23.6261	–7.311	–	–	3.0733	–
	Timoshenko and Goodier [13]	25.4896	0	3.0000	0	–	–
10	Present method	154.0091	14.255	7.6519	2.025	7.5733	–
	Reddy [12]	154.0091	14.255	7.6519	2.025	7.5733	–
	Timoshenko [5]	147.6634	–0.041	5.1223	–31.702	7.6834	–
	Euler-Bernoulli	147.6630	–0.0412	–	–	7.6834	–
	Timoshenko and Goodier [13]	147.7239	0	7.500	0	–	–

TABLE V. NON-DIMENSIONAL AXIAL DISPLACEMENT $\bar{u}(x = l, z = h/2)$ AND TRANSVERSE DISPLACEMENT $\bar{w}(x = l/2, z = 0)$ FOR THICK ISOTROPIC BEAMS SUBJECTED TO LINEARLY DISTRIBUTED LOAD FOR VARIOUS VALUES OF ASPECT RATIOS.

l/h	Theory/Reference	$\bar{u}$	% Diff	$\bar{w}$	% Diff
2	Present method	1.225	2.045	1.2660	3.2205
	Reddy [12]	1.225	2.045	1.2660	3.2205
	Timoshenko [5]	1.0000	–9.091	1.2690	3.465
	Euler-Bernoulli	1.0000	–9.091	0.7815	–36.282
	Timoshenko and Goodier [13]	1.1000	0	1.2265	0
4	Present method	8.2520	4.456	0.9030	1.1765
	Reddy [12]	8.2520	4.456	0.9030	1.1765
	Timoshenko [5]	8.0000	1.266	0.9030	1.1765
	Euler-Bernoulli	8.0000	1.266	0.7815	–12.437
	Timoshenko and Goodier [13]	7.9000	0	0.8925	0
10	Present method	125.635	0.709	0.8010	0.250
	Reddy [12]	125.635	0.709	0.8010	0.250
	Timoshenko [5]	125.000	0.200	0.8010	0.250
	Euler-Bernoulli	125.000	0.200	0.7815	–2.190
	Timoshenko and Goodier [13]	124.750	0	0.7990	0

TABLE VI. DIMENSIONLESS AXIAL BENDING STRESSES $\bar{\sigma}_{xx}(x = l/2, z = h/2)$, TRANSVERSE SHEAR STRESSES $\bar{\tau}_{xz}^{CR}$ $(x = 0, z = 0)$, $\bar{\tau}_{xz}^{EE}(x = 0, z = 0)$ FOR THICK ISOTROPIC BEAMS SUBJECTED TO LINEARLY DISTRIBUTED LOAD FOR VARIOUS VALUES OF ASPECT RATIOS.

l/h	Theory/Reference	$\bar{\sigma}_{xx}$	% Diff	τ_{xz}^{CR}	% Diff	τ_{xz}^{EE}	% Diff
2	Present method	1.6310	1.938	0.7075	–5.667	0.6310	–
	Reddy [12]	1.6310	1.938	0.7075	–5.667	0.6310	–
	Timoshenko [5]	1.500	–6.25	0.600	–20	0.7385	–
	Euler-Bernoulli	1.500	–6.25	–	–	0.7500	–
	Timoshenko and Goodier [13]	1.600	0	0.7500	0	–	
4	Present method	6.1315	0.5164	1.4540	–3.067	1.3975	–
	Reddy [12]	6.1315	0.5164	1.4540	–3.067	1.3975	–
	Timoshenko [5]	6.000	–1.6393	1.2000	–20	1.4765	–
	Euler-Bernoulli	6.000	–1.6393	–	–	1.4765	–
	Timoshenko and Goodier [13]	6.100	0	1.500	0	–	–
10	Present method	37.634	0.090	3.6805	–1.853	3.6520	–
	Reddy [12]	37.634	0.090	3.6805	–1.853	3.6520	–
	Timoshenko [5]	37.500	–0.266	3.000	–20	3.6915	–
	Euler-Bernoulli	37.500	–0.266	–	–	3.6915	–
	Timoshenko and Goodier [13]	37.600	0	3.750	0	–	–

V. DISCUSSION

Tables I and Table II present the results for displacements and stresses in thick beams carrying uniformly distributed loads. Table 1 shows that the present method gives identical results with previous results obtained using the Ritz variational method by Ike [14] and for $l/h = 2$ (corresponding to thick beams), the present results for transverse deflection which is 3.221% greater than the Timoshenko and Goodier exact solution is better than the Timoshenko solution which is 3.465% greater than the exact solution. For l/h greater than 2, which is for thin and moderately thick beams, the present results for $w(x = l/2, z = 0)$ are the same as the Timoshenko results. Similarly, from Table II, the results for $\sigma_{xx}(x = l/2, z = h/2)$ show a relative difference ranging from 0.090% for $l/h = 10$ to 1.960% for $l/h = 2$. The table further reveals that the results for σ_{xx} is better estimated using the present theory than the Timoshenko theory which yields a relative difference of –6.25% for $l/h = 2.0$.

The results for moderately thick beams under point load at midspan are displayed in Table 3 for displacements and Table IV for stresses. Table III shows that the transverse displacements $\bar{w}(x = l/2, z = 0)$ from the TSDT give better results than the Timoshenko theory with variations from the exact results of Timoshenko and Goodier varying from 0.292% for l/h = 10 to 7.899% for l/h = 2 as compared with Timoshenko's results that present variations of 0.304% for l/h = 10 to 9.978% for l/h = 2.

Table IV shows that for σ_{xx}, the present TSDT shows greater relative difference varying from 14.255% for l/h = 10 to 68.571% for l/h = 2 as compared with Timoshenko's FSDT. The unacceptable results obtained may be due to the singularity introduced by the point load. However, acceptable results are obtained for the transverse shear stresses τ_{xz} where the relative difference for the exact solution is less than 3.10% for all cases of l/h considered.

Tables V and Table VI present respectively the displacements and stresses in moderately thick beams subjected to linearly distributed load to various l/h. The tables show that the present Fourier series results for TSDT are identical with previous results obtained by Reddy. Table V further shows that the present results for $w(x = l/2, z = 0)$ vary from 0.25% for l/h = 10 to 3.2205% for l/h = 2 from the exact elasticity result of Timoshenko and Goodier. The variations show that the TSDT presents more accurate results for beams than the TBT as the TBT gives higher relative differences for $w(x = l/2, z = 0)$. The relative differences for $\sigma_{xx}(x = l/2, z = h/2)$ are 0.090% for l/h = 10 and 1.935% for l/h = 2, which are lower than the relative differences calculated for Timoshenko's results; illustrating that the present theory is more accurate than the TBT. Table IV further shows that the Fourier series solution presented in this paper are identical with the Reddy third order shear deformable beam solution by classical methods.

VI. Conclusion

In conclusion,

(i) The axial displacement obtained is an infinite cosine series.

(ii) The expression for the maximum transverse displacement is an infinite sine series and the maximum value of w is found at $x = l/2$, $z = 0$.

(iii) The axial flexural stress σ_{xx} is an infinite sine series expression with maximum values at $x = l/2$, $z = \pm h/2$.

(iv) The transverse shear stress expression is an infinite cosine series, with maximum values at $x = 0$, $z = 0$.

(v) The third order shear deformation theory (TSDT) of beams is consistent with the theory of elasticity formulation principles. The boundaries $z = \pm h/2$ are free of transverse shear stresses and strains.

References

[1] C.C. Ike and E.U. Ikwueze, "Ritz method for the analysis of statically indeterminate Euler-Bernoulli beams," Saudi Journal of Engineering and Technology, Vol. 3, Issue 3, pp. 133 – 140, 2018.

[2] C.C. Ike and E.U. Ikwueze, "Fifth degree Hermittian polynomial shape functions for the finite element analysis of clamped simply supported Euler-Bernoulli beam," American Journal of Engineering Research, Vol. 7, Issue 4, pp. 97 – 105, 2018.

[3] C.C. Ike, "Fourier sine transform method for the free vibration of Euler-Bernoulli beam resting on Winkler foundation," International Journal of Darshan Institute on Engineering Research and Emerging Technologies (IJDI-ERET), Vol. 7, No. 1, pp. 1 – 6, 2018.

[4] C.C. Ike, "Point collocation method for the analysis of Euler-Bernoulli beam on Winkler foundation," International Journal of Darshan Institute on Engineering Research and Emerging Technologies (IJDI-ERET), Vol. 7, No. 2, pp 1 – 7, 2018.

[5] S.P. Timoshenko, "On the correction for shear of the differential equation for transverse vibration of prismatic bars," Philosophical Magazine, Vol. 41 No. 6, pp. 742 – 746, 1921.

[6] M. Levinson, "A new rectangular beam theory," Journal of Sound and Vibration, Vol. 74 No. 1, pp. 81 – 87, 1981.

[7] R.P. Shimpi, P.J. Guruprasad, and K.S. Pakhare, "Simple two variable refined theory for shear deformable isotropic rectangular beams," Journal of Applied and Computational Mechanics, Vol. 6 No. 3, pp. 394 – 415, 2020.

[8] Y.M. Ghugal and R. Sharma, "A refined shear deformation theory for flexure of thick beams," Latin American Journal of Solids and Structures, Vol. 8, pp. 183 – 195, 2011.

[9] S.A. Ambartsumyan, "On the theory of bending of plates," Izv Otd Tech Nauk ANSSSR, Vol. 5, pp. 67 – 77, 1958.

[10] E.T. Kruszewski, Effect of transverse shear and rotary inertia on the natural frequency of a uniform beam. National Advisory Committee for Aeronautics Technical Note 1909 (NACA-TN-1909), pp. 1 – 16, 1949.

[11] S.S. Akavci, "Buckling and free vibration analysis of symmetric and antisymmetric laminated composite plates on an elastic foundation," Journal of Reinforced Plastics and Composites, Vol. 26 No 18, pp. 1907 – 1919, 2007.

[12] J.N. Reddy, "A general non-linear third order theory of plates with moderate thickness," International Journal of Non-linear Mechanics, Vol. 25 No. 6, pp. 677 – 686, 1990.

[13] S.P. Timoshenko and J.N. Goodier, Theory of Elasticity, 3rd International Edition. Singapore: McGraw Hill, 1970.

[14] C.C. Ike, "Ritz variational method for the flexural analysis of third order shear deformable beams," Conference paper presented at Conference on Engineering Research Technology Innovation and Practice (CERTIP). Faculty of Engineering, University of Nigeria, Nsukka, 3rd – 6th November, 2020.

[15] C.C. Ike and O.A. Oguaghamba, "Trigonometric shear deformation theory for the bending analysis of thick beams: Fourier series method," Conference paper presented at Conference on Engineering Research Technology Innovation and Practice (CERTIP). Faculty of Engineering, University of Nigeria, Nsukka, 3rd – 6th November, 2020.

[16] C.C. Ike, Timoshenko beam theory for the flexural analysis of moderately thick beams – variational formulation and closed form solutions. Tecnica Italiana – Italian Journal of Engineering Science, Vol. 63 No. 1, pp. 34 – 45, 2019.

[17] H.N. Onah, C.U. Nwoji, M.E. Onyia, B.O. Mama, and C.C. Ike, "Exact solutions for the elastic buckling problem of moderately thick beams," Revue des Composites et des Materiaux Avances, Vol. 30, No. 2, pp. 83 – 93, 2020.

[18] C.C. Ike, C.U. Nwoji, B.O. Mama, H.N. Onah, and M.E. Onyia, "Laplace transform method for the elastic buckling analysis of moderately thick beams," International Journal of Engineering Research and Technology, Vol. 12, No. 10, pp. 1626 – 1638, 2019.

[19] Y. Ghugal, A two-dimensional exact elasticity solution of thick beams. Departmental Report – 1. Department of Applied Mechanics, Government Engineering College, Aurangabad, India, pp 1 – 96, 2006.

[20] Y.M. Ghugal and R.P. Shimpi, "A review of refined shear deformation theories for isotropic and anisotropic laminated beams," Journal of Reinforced Plastics and Composites, Vol. 20 No. 3, pp. 255 – 272, 2001.

[21] P.R. Heyliger and J.N. Reddy, "A higher order beam finite element for bending and vibration problems," Journal of Sound and Vibration, Vol. 126 No. 2, pp. 309 – 326, 1988.

[22] Y.M. Ghugal and A.G. Dahake, "Flexural analysis of deep beam subjected to parabolic loads using refined shear deformation theory," Applied Computational Mechanics, Vol 6 No. 2, pp 163 – 172, 2012.

[23] A.S. Sayyad, "Comparison of various shear deformation theories for the free vibration of thick isotropic beams," Latin American Journal of Solids and Structures, Vol. 2 No 1, pp. 85 – 97, 2011.

[24] A.S. Sayyad, "Comparison of various refined beam theories for the bending and free vibration analysis of thick beams," Applied and Computational Mechanics, Vol. 5, pp. 217 – 230, 2011.