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Galerkin Kantorovich Method for Solving the Terzaghi's One-Dimensional Soil Consolidation Equation

Charles Chinwuba Ike[1]*, Benjamin Okwudili Mama[2], Onyedikachi Aloysius Oguaghamba[2], Michael E. Onyia[2]

[1]*Department of Civil Engineering, Enugu State University of Science and Technology, Agbani, Enugu State, Nigeria*, *charles.ike@esut.edu.ng*

[2]*Department of Civil Engineering, University of Nigeria, Nsukka, Enugu State, Nigeria*, (*benjamin.mama@unn.edu.ng*, *aloysius.oguaghamba@unn.edu*, *michael.onyia@unn.edu.ng)*

Abstract

This paper presents the Galerkin-Kantorovich variational method for solving the Terzaghi's one-dimensional consolidation equation for two-way drainage conditions. The solution was considered as an infinite series of known coordinate (shape) functions and unknown function $\varphi(t)$ of time which we sought such that the resulting functional is minimized. The shape functions satisfied the hydraulic boundary conditions at the boundary of the consolidating soil. Galerkin-Kantorovich variational integral equation was thus formulated for the initial boundary value problem using residual minimization principles. The solution resulted in a system of first order ordinary differential equations in $\varphi_n(t)$ which was solved for $\varphi_n(t)$. Orthogonalization principles were used to obtain the integration constants in terms of initial pore water pressure, thus yielding the general solution. Solutions for constant initial excess pore water pressure were obtained and found to be the closed-form solution. The solutions were presented in terms of global (average) degrees of consolidation and tabulated. The results obtained were exact and identical with results previously found using separation of variables techniques.

Keywords: *Galerkin-Kantorovich Method, Terzaghi's One-Dimensional Consolidation Equation, Excess Pore Water Pressure Distribution, Average Degree of Consolidation*

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I. INTRODUCTION

Consolidation is the time dependent process of dissipation of pore water pressures in saturated and partially saturated soils with low coefficients of hydraulic permeabilities due to external loads applied to the soil [1 – 3]. The process is important in foundation engineering analysis and design because of the resulting consolidation settlement which is time-dependent.

There are many types of consolidation associated with the types of transient seepage namely: one-dimensional, two-dimensional, three-dimensional, radial, and axisymmetric consolidation. Accordingly, various theories of soil consolidation have been presented. Some are:

(i) Terzaghi's one-dimensional (1D) consolidation theory
(ii) Two-dimensional (2D) consolidation theory
(iii) Three-dimensional (3D) consolidation theory
(iv) Gibson's theory [4]
(v) Rendulic's radial consolidation theory
(vi) Biot's consolidation theory
(vii) Barron's three-dimensional soil consolidation theory [5]

The fundamental laws that are used in deriving the governing theories of the consolidation process are: Darcy's law or the law of seepage flow, the continuity law and the soil mineral constitutive laws. Literature review shows that the soil consolidation problems have been solved using both analytical and numerical techniques.

Analytical techniques that have been used are methods commonly used for solving partial differential equations (PDEs) and are the methods of separation of variables (product method) and eigenfunction expansion (Fourier series method) [6].

Numerical techniques used for soil consolidation analysis are commonly used numerical methods for solving PDEs and they include finite element method (FEM), finite difference method (FDM), boundary element method (BEM) and differential quadrature method (DQM).

Axisymmetric consolidation problems have been investigated by Leo [7], Barron [5], Ho et al [8], Zhou [9], Zhou and Tu [10], Conte [11], and Shi and Zhang [12].

Two-dimensional consolidation problems have been studied by Conte [11], and Ho et al [13].

One-dimensional consolidation problems have been studied for various assumptions of soil stress-strain behaviours, degrees of saturation, layering of soil and nature of applied loading by Zhou and Zhao [14]; Shan et al [15]; Zhou et al [16]; Ma et al [17]; Olek [3, 18]; Zhang et al [19]; Conte and Troncone [20]; Gibson [4]; and Ike [1].

Cao et al [21] have studied large-strain consolidation of soil. Radhika et al [22] have presented a review on consolidation theories and their applications. Further studies of soil consolidation are presented in Das [23] and Craig [24]. Wang et al [25] used elementary functions to express a simplified solution to one-dimensional consolidation with a threshold gradient. They obtained approximate solutions that violated the field equation but satisfied the boundary conditions.

McDonald et al [26] solved Terzaghi's one-dimensional (1-D) soil consolidation equation using a finite difference method (FDM) and Microsoft Excel spreadsheet, a readily available computational tool. Their work used Microsoft Visual Basic Application (VBA) tool in Excel to write and run finite difference analysis routines of the 1-D soil consolidation equation.

Zhang et al [27] used the Laplace transformation method to obtain classical solutions to the Terzaghi one-dimensional soil consolidation problem for impermeable bottom boundary and simplified assumption for initial and boundary conditions.

This paper applies the Galerkin-Kantorovich method to solve the Terzaghi's one-dimensional consolidation equation for two way drainage conditions and constant initial excess pore water pressure distribution.

II. Governing Equation Of Terzaghi's 1D Consolidation Equation

The partial differential equation for 1D consolidation in the z direction is [1, 3]:

$$c_z \frac{\partial^2 u_e(z,t)}{\partial z^2} = \frac{\partial u_e(z,t)}{\partial t} \tag{1}$$

where $c_z = \dfrac{k_z}{m_v \gamma_w}$ (2)

c_z is the coefficient of 1D consolidation in the z direction, k_z is the hydraulic permeability coefficient in the z coordinate direction, m_v is the volume compressibility coefficient, γ_w is the weight density of water, $u_e(z, t)$ is the excess pore water pressure variation with depth and time in the consolidating soil, t is time, z is the depth coordinate.

A. Problem considered – consolidation of clay soil bounded by two layers of permeable soil

The paper considers the consolidation under double-drainage conditions shown in Figure 1.

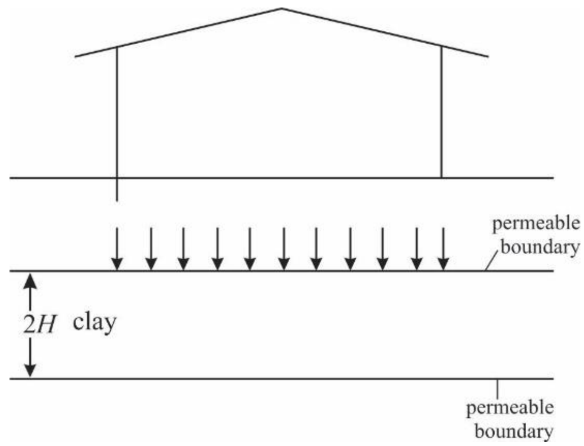

Fig. 1. Clay soil consolidating under uniform pressure and double drainage conditions.

The initial conditions are

$$u_e(z, t=0) = u_e(z, 0) \tag{3}$$

for $0 \le z \le 2H$

wherein $u_e(z,0)$ is the known initial pore water pressure distribution at the start of the consolidation process.

The boundary conditions are:

$$u_e(z=0, t) = 0 \quad \text{for } t > 0,\ t \to \infty \tag{4}$$

$$u_e(z=2H, t) = 0 \quad \text{for } t > 0,\ t \to \infty \tag{5}$$

III. Galerkin Method

The Galerkin method is a numerical method that seeks an approximate solution for the unknown function u to a differential equation of the form:

$$Lu = p \tag{6}$$

where L is the differential operator and p is forcing function.

The approximation to u is sought in terms of a linear combination of coordinate (basis) functions N_i as:

$$\bar{u} = \sum_{i=1}^{n} N_i u_i \tag{7}$$

and u_i are the unknown quantities that need to be found, $\bar{u}$ is approximation to u, n is the number of coordinate functions used in the approximation.

Substitution of the approximation for u given by Equation (2) in Equation (1) gives an error function, $\bar{e}$:

$$L\bar{u} - p = \bar{e} \tag{8}$$

The Galerkin method equation is built on the assumption that the weighted average error of the approximation should be zero.

This yields:

$$\iiint_v \bar{e}\, w\, dv = \iiint_v (L\bar{u} - p) w\, dv \tag{9}$$

where w is the weighting function, v denotes the domain of integration.

$$\iiint_v (L \sum N_i u_i - p) w)\, dv \tag{10}$$

$$\sum \iiint_v L N_i u_i\, w\, dv = \iiint_v p w\, dv \tag{11}$$

The weight function can be approximated using the same coordinate functions as the assumed approximation.

The Galerkin equations become the system of equations:

$$\sum \iiint_v L N_i u_i\, N_j\, dv = \iiint_v p N_j\, dv \tag{12}$$

$i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, n.$

IV. Galerkin-Kantorovich Variational Integral Formulation And General Solution Of The Problem

By the Galerkin-Kantorovich variational method, the solution for the unknown excess pore water pressure distribution $u_e(z, t)$ in the governing PDE is assumed in terms of an infinite series of coordinate (shape) functions that satisfy the boundary conditions at $z = 0$, and $z = 2H$ and unknown functions of time. Thus,

$$u_e(z,t) = \sum_{n=1}^{\infty} \varphi_n(t) \phi_n(z) \tag{13}$$

where $\phi_n(z)$ is the coordinate (shape) function that satisfies the boundary conditions in Equations (4) and (5).

Hence, $\phi_n(z) = \sin \frac{n\pi z}{2H}$ (14)

The Galerkin-Kantorovich variational integral equation for the problem is constructed from:

$$\int_0^{2H} R(z,t) \phi_n(z) dz = 0 \tag{15}$$

where $R(z, t)$ is the Residual function.

Thus,

$$\int_0^{2H} \left(-\sum_{n=1}^{\infty} \left(\frac{d\varphi_n(t)}{dt} + c_z \left(\frac{n\pi}{2H} \right)^2 \varphi(t) \right) \sin \frac{n\pi z}{2H} \right) \sin \frac{m\pi z}{2H} dz = 0$$

…(16)

Simplifying,

$$-\sum_{n=1}^{\infty} \left(\frac{d\varphi_n(t)}{dt} + c_z \left(\frac{n\pi}{2H} \right)^2 \varphi(t) \right) * \int_0^{2H} \sin \frac{n\pi z}{2H} \sin \frac{m\pi z}{2H} dz = 0$$

…(17)

From the orthogonality property of $\sin \frac{n\pi z}{2H}$,

$$I_{nm} = \int_0^{2H} \sin \frac{n\pi z}{2H} \sin \frac{m\pi z}{2H} dz = 0 \qquad \text{if } n \neq m \tag{18}$$

$$I_{nm} = \int_0^{2H} \sin \frac{n\pi z}{2H} \sin \frac{m\pi z}{2H} dz = H \qquad \text{if } n = m \tag{19}$$

Hence the condition for nontrivial solutions of Equation (17) is the first order linear differential equation:

$$\frac{d\varphi_n(t)}{dt} + c_z \left(\frac{n\pi}{2H} \right) \varphi_n(t) = 0 \tag{20}$$

Solving using the classical methods for solving first order differential equation gives:

$$\varphi_n(t) = a_n \exp\left(-\left(\frac{n\pi}{2H} \right)^2 c_z t \right) \tag{21}$$

where a_n is the integration constant.

The general solution is then obtained as:

$$u_e(z,t) = \sum_{n=1}^{\infty} a_n \exp\left(-\left(\frac{n\pi}{2H} \right)^2 c_z t \right) \sin \frac{n\pi z}{2H} \tag{22}$$

V. Results

A. General Results for $u_e(z, 0)$

In order to obtain results for known initial excess pore water pressure variations, a_n needs to be evaluated. For $t = 0$, in Equation (22)

$$u_e(z, t=0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{2H} = u_e(z, 0) \tag{23}$$

Multiplying both sides of Equation (23) by $\sin \frac{m\pi z}{2H}$ and integrating with respect to z over the domain of the consolidation soil, we obtain a_n from this orthogonalization process in Fourier series theory.

$$\int_0^{2H} u_e(z,0) \sin \frac{m\pi z}{2H} dz = \int_0^{2H} \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{2H} \sin \frac{m\pi z}{2H} dz \tag{24}$$

$n \neq m$ gives trivial solutions while $n = m$ gives nontrivial solutions as:

$$\int_0^{2H} u_e(z,0) \sin \frac{n\pi z}{2H} dz = \sum_{n=1}^{\infty} a_n \int_0^{2H} \sin^2 \frac{n\pi z}{2H} dz = \sum_{n=1}^{\infty} a_n H \tag{25}$$

$$a_n = \frac{1}{H} \int_0^{2H} u_e(z,0) \sin \frac{n\pi z}{2H} dz \tag{26}$$

The general solution when the $u_e(z, 0)$ is given then becomes:

$$u_e(z,t) = \sum_{n=1}^{\infty} \left(\frac{1}{H} \int_0^{2H} u_e(z,0) \sin \frac{n\pi z}{2H} dz \right) \sin \frac{n\pi z}{2H} \exp\left(-\left(\frac{n\pi}{2H} \right)^2 c_z t \right)$$

…(27)

The analytical expression obtained for $u_e(z, t)$ agrees perfectly well with other expressions previously obtained by other scholars using the method of separation of variables. The result is identical with closed form expressions.

B. Result For Constant Initial Excess Pore Water Pressure Variation In The Consolidating Soil

Here, $u_e(z,0) = u_0$ (28)

Then, by substitution of $u_e(z, 0)$ in Equation (19)

$$u_e(z,t) = \sum_{n=1}^{\infty}\left(\frac{1}{H}\int_0^{2H} u_0 \sin\frac{n\pi z}{2H}dz\right)\sin\frac{n\pi z}{2H}\exp\left(-\left(\frac{n\pi}{2H}\right)^2 c_z t\right) \quad \ldots(29)$$

Evaluating the integral and simplifying,

$$u_e(z,t) = \sum_{n=1}^{\infty}\frac{2u_0}{n\pi}(1-\cos n\pi)\sin\frac{n\pi z}{2H}\exp\left(-\left(\frac{n\pi}{2H}\right)^2 c_z t\right) \quad (30)$$

Let $N = \frac{n\pi}{2}$ (31a)

and $\frac{c_z t}{H^2} = T_v$ (31b)

Then,

$$u_e(z,t) = \sum_{n=1}^{\infty}\frac{2u_0}{N}\sin\frac{Nz}{H}\exp(-N^2T_v) \quad (32)$$

Local degree of consolidation, U_z

This is obtained from

$$U_z = 1 - \frac{u_e(z,t)}{u_0} \quad (33)$$

Thus,

$$U_z = 1 - \sum_{n=1}^{\infty}\frac{2}{N}\sin\frac{Nz}{H}\exp(-N^2T_v) \quad (34)$$

Average (global) degree of consolidation, U

This is obtained from

$$U = 1 - \frac{\frac{1}{2H}\int_0^{2H} u_e(z,t)dz}{u_0} \quad (35)$$

Thus,

$$U = 1 - \sum_{n=1}^{\infty}\frac{2}{N^2}\exp(-N^2T_v) \quad (36)$$

Table 1 shows the values of U for assumed values of T_v for the two-way drainage of 1D consolidating soil for the case of constant initial excess pore water pressure variation. The table shows the obtained results agree with previous results by Ike [1] and other scholars.

TABLE I. TIME FACTOR T_v – AVERAGE DEGREE OF CONSOLIDATION U TABLE FOR 1D CONSOLIDATION WITH DOUBLE DRAINAGE AND CONSTANT INITIAL PORE WATER PRESSURE

U	T_v (present study)	T_v (Ike [1])
0	0	0
5	0.00196	0.00196
10	0.00785	0.00785
15	0.0177	0.0177
20	0.0314	0.0314
25	0.0491	0.0491
30	0.0707	0.0707
35	0.0962	0.0962
40	0.1257	0.1257
45	0.159	0.159
50	0.196	0.197
55	0.239	0.239
60	0.2827	0.2827
65	0.3404	0.3404
70	0.4028	0.4028
75	0.4767	0.4767
80	0.5671	0.5671
85	0.6837	0.6837
90	0.848	0.848
95	1.129	1.129
100	∞	∞

VI. DISCUSSION

This paper has presented the Galerkin-Kantorovich variational method for solving the Terzaghi 1D soil consolidation equation for two-way drainage and constant initial excess pore water pressure variation. The governing PDE was expressed in variational form using weighted residual techniques.

Solving the variational formulation resulted in a first order ODE which was solved to obtain $\varphi_n(t)$ as the exponential decay function of time given in Equation (21). The general solution to the PDE is found as Equation (22). The general solution to the PDE is found in terms of the initial pore water pressure $u_e(z, 0)$ as Equation (27).

Equation (27) is a closed form expression for the solution since the governing equation is satisfied identically at all points $0 \le z \le 2H$ in the consolidating soil as well as on the boundaries z = 0, and z = 2H. The obtained expression – Equation (27) – is identical with expression obtained earlier by Ike [1], Das [23], and Craig [24].

The solution for constant initial excess pore water pressure is obtained in dimensionless form as Equation (32). The solution for constant initial excess pore water pressure is presented in terms of average degree of consolidation as Equation (36).

Equation (36) is presented as Table 1 which also presents the results obtained by Ike [1]. Table 1 shows the present results for U vs T_v agree identically with previous results obtained using other techniques by other researchers.

VII. CONCLUSION

In conclusion,

(i) The Galerkin-Kantorovich variational method converts the Terzaghi 1D consolidation problem which is a boundary value problem to an integral equation.

(ii) The present method gave exact solutions to the problems since the exact shape functions were used for the unknown $\phi_n(z)$.

(iii) The general solution obtained for $u_e(z, t)$ in terms of $u_e(z, 0)$ could be used to find solutions for $u_e(z, t)$ once the mathematical function for $u_e(z, t)$ is given or known.

(iv) The method gave exact mathematical solutions for $u_e(z, t)$ in the boundary value problem (BVP) solved, and the exact solutions for the average degree of consolidation.

(v) The results obtained are identical to previous results.

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